**Cyclic inequality involving square roots**

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Let $a, b$ and $c$ be positive real numbers. Prove that

$$\frac{2a}{\sqrt{3a+b}} + \frac{2b}{\sqrt{3b+c}} + \frac{2c}{\sqrt{3c+a}} \le \sqrt{3(a+b+c)}.$$

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First we will prove the following auxiliary inequality:

(**1**) $$\sum \frac{a}{3a+b} \le \frac{3}{4}.$$

Combining Cauchy Inequality and inequality $ab+bc+ca \le \dfrac{(a+b+c)^2}{3}$

we obtain $\sum \dfrac{b}{3a+b} = \sum \dfrac{b^2}{3ab+b^2} \ge \dfrac{(a+b+c)^2}{\sum(3ab+b^2)} =$

$$\frac{(a+b+c)^2}{(a+b+c)^2+(ab+bc+ca)} \ge \frac{(a+b+c)^2}{(a+b+c)^2+(a+b+c)^2/3} = \frac{3}{4}.$$

Hence, $\sum \dfrac{3a}{3a+b} = \sum\left(1 - \dfrac{b}{3a+b}\right) = 3 - \sum \dfrac{b}{3a+b} \le 3 - \dfrac{3}{4} = \dfrac{9}{4} \Leftrightarrow$(**1**)

Using inequality (**1**) and again Cauchy inequality we obtain

$$\sum \frac{a}{\sqrt{3a+b}} = \sum \sqrt{a} \cdot \sqrt{\frac{a}{3a+b}} \le \sqrt{a+b+c} \cdot \sqrt{\sum \frac{a}{3a+b}} \le \sqrt{a+b+c} \cdot \frac{\sqrt{3}}{2} \Leftrightarrow$$

$$\sum \frac{2a}{\sqrt{3a+b}} \le \sqrt{3(a+b+c)}.$$